Lecture Note 2: A "Crash Course" in Optimization

ECN 100B: Intermediate Microeconomic Theory

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The big picture

- Economic analysis usually involves two basic steps:
  - Predict how a person or firm will behave in a certain situation
  - Predict how behavior changes depending on the circumstances
- Today, we'll develop the basic tools for each of these steps:
  - To predict behavior, we solve an **optimization problem**
  - To see how behavior changes, we do **comparative statics**

1. Overview

Some terminology

- **objective function**: what I want to maximize (or minimize)
  - For consumers: utility function; for firms: profit function
  - Gives a "score" to every option available
- **choice variable**: whatever I get to choose
  - How much output to produce, whether to go skiing
  - Sometimes there is one choice variable, sometimes more
- **choice set**: the list of available options
  - Output: any non-negative real number
  - Skiing: yes or no
- **parameters**: outside factors that I take as "given"
  - Output: market price (if I'm a price-taker)
  - Skiing: temperature, snowfall

Optimization with a finite number of options

- Sometimes there are only a few options
  - Should I buy trip insurance for my next flight?
  - Should Apple open a store in downtown Davis?
- In these cases:
  - Evaluate the objective function for each option
  - Choose the option with the best "score"
  - If there are ties, the decision-maker is indifferent
- We'll see examples of this soon

Optimization with an infinite number of options

- But usually we study problems with infinitely many options
  - How many units to produce
  - How many hours to work
- In these cases:
  - Write the objective function in terms of the choice variable
  - Use calculus to find the max (or min) of the objective function

2. Writing Down the Problem

Example: soybean production

- Let's revisit a problem from last lecture:
  - Soybean farm in a competitive market
  - Faces (constant) price $p = 20$
  - Has cost function $C(q) = q^2$
  - How much should it produce?

- Starting point: write down the optimization problem

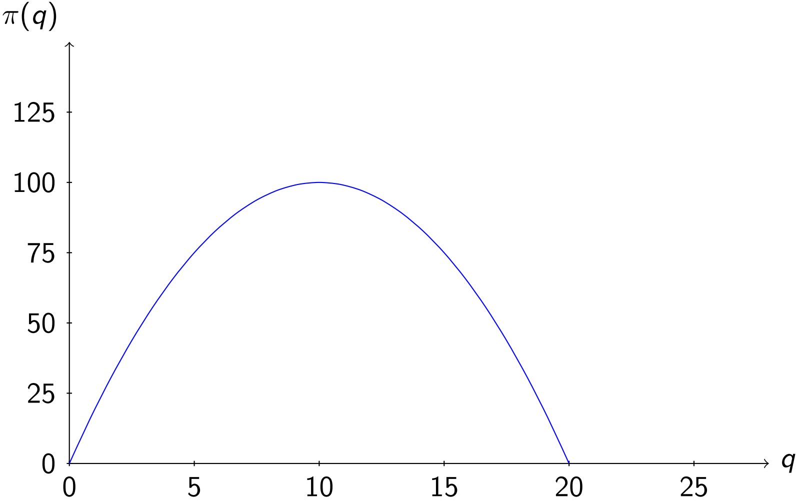
$$\max_{q \geq 0} \pi(q) = \underbrace{20q}_{\text{revenue}} - \underbrace{q^2}_{\text{cost}}$$

- "$\max\limits_{q \geq 0}$" means "choose a non-negative value of q"
  - Throughout ECN 100B, quantities can never be negative
  - For simplicity, however, I will usually just write "$\max\limits_{q}$"

3. Finding a Candidate Optimum

Finding the “top of the hill”

Equation: $\pi(q) = 20q - q^2$



First-order conditions (FOCs)

- To figure out where profits are maximized, we take the FOC:
  - Differentiate the objective function by the choice variable
  - Treat everything else as a constant
  - Set the (first) derivative equal to zero

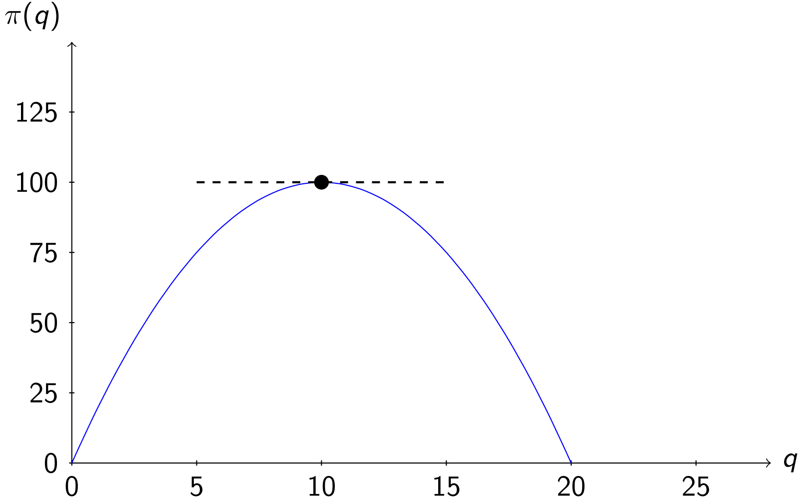
- So, to solve: $\max_q \ \pi(q) = 20q - q^2$

$$\frac{d}{dq}\pi(q) = \underbrace{20 - 2q}_{\text{marginal profit}} = 0 \implies q^* = 10$$

- The FOC finds points where the objective function is "flat"

Finding the “top of the hill”

Equation: $\pi(q) = 20q - q^2$

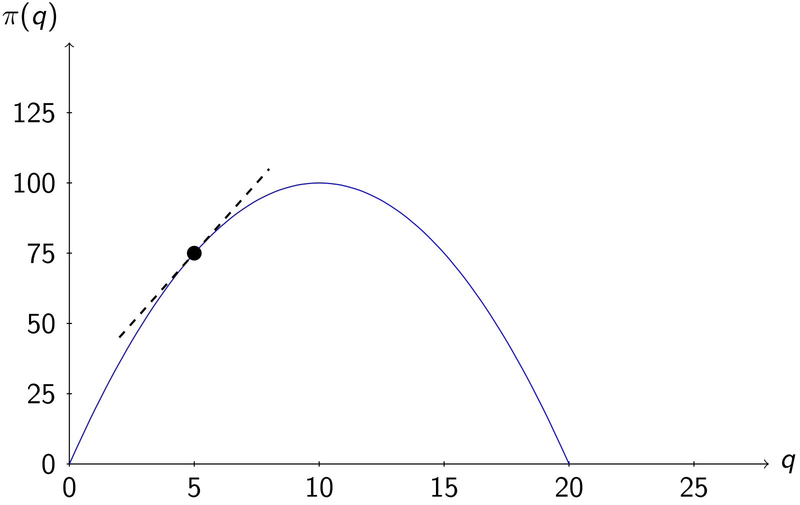


An economic perspective

- π is maximized when there is **no profitable deviation**
  - Profitable deviation: a change in behavior that increases profits
- Why can't $q = 5$ be the optimal choice?
  - Marginal revenue $= 20$
  - Marginal cost $= 2 \times 5 = 10$
  - $\implies$ Marginal profit $= 20 - 10 = 10$
  - $\implies$ We can increase π by increasing q
- Why can't $q = 15$ be optimal?
  - Marginal revenue $= 20$
  - Marginal cost $= 2 \times 15 = 30$
  - $\implies$ Marginal profit $= 20 - 30 = -10$
  - $\implies$ We can increase π by *decreasing* q
- At the optimum, such improvements are impossible

"Climbing up the hill"

Equation: $\pi(q) = 20q - q^2$



"Climbing up the hill"

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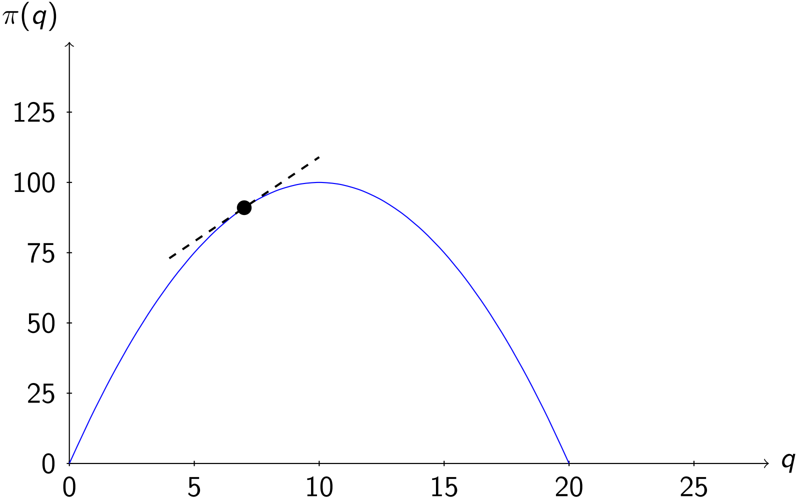

"Climbing up the hill"

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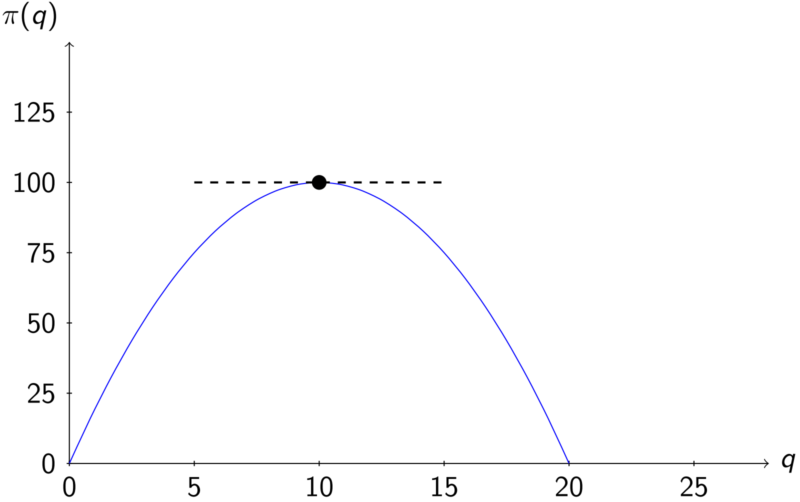


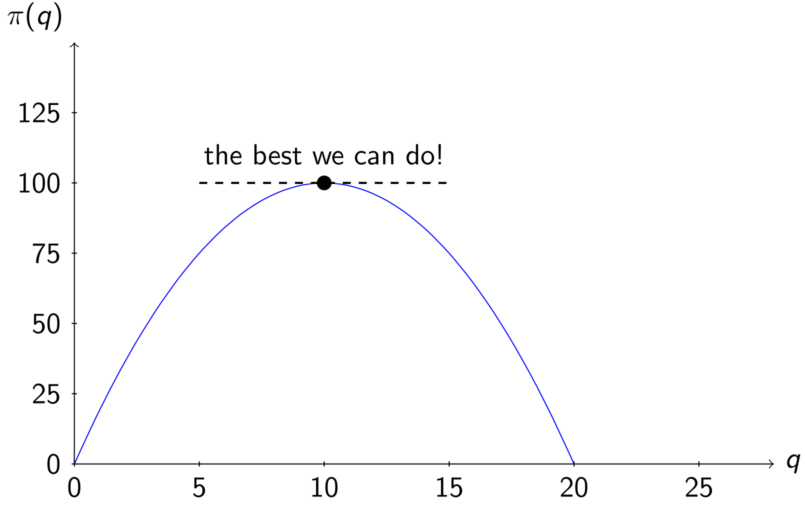
4. Verifying a Candidate Optimum

Wait: is it really the right answer?

- Solving the FOC doesn't <u>always</u> get us the right answer
  - Sometimes it gives us a *minimum*, not a *maximum*
  - And sometimes the right answer is a "corner solution"
- Have to verify that our "candidate optimum" is really optimal

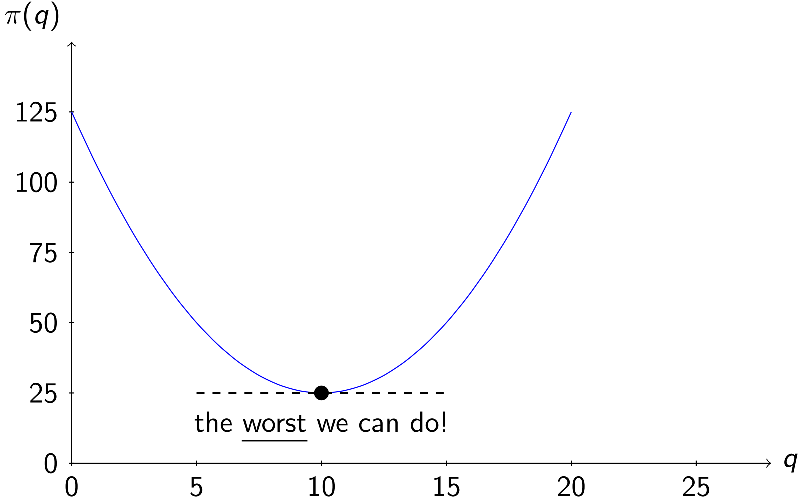
Here, $q = 10$ is the global maximum

Equation: $\pi(q) = 20q - q^2$



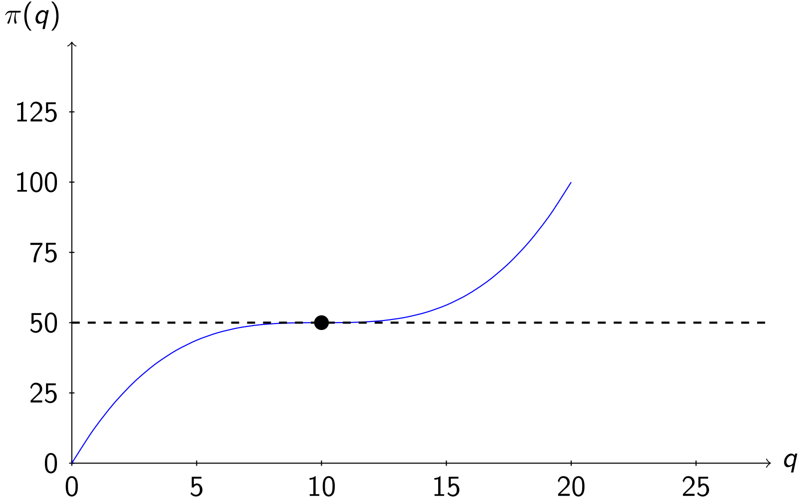
But here, $q = 10$ is the global minimum

Equation: $\pi(q) = q^2 - 20q + 125$



And here, $q = 10$ is a saddle point (neither max nor min)

Equation: $\pi(q) = \frac{1}{20}(q-10)^3 + 50$



The second-order condition

- How can we tell if a candidate maximum is really a maximum?
- Check the second derivative:
  - If $\frac{d^2\pi(q)}{dq^2} < 0$ at $q = q^*$: locally concave $\implies$ local maximum
  - If $\frac{d^2\pi(q)}{dq^2} < 0$ for all q: globally concave $\implies$ global maximum
- This is called the "second-order condition" (or SOC)
  - If we're looking for a minimum, we need $\frac{d^2\pi(q)}{dq^2} > 0$
- For $\pi(q) = 20q - q^2$:

$$\frac{d^2\pi(q)}{dq^2} = -2 < 0 \implies q^* = 10 \text{ is a global maximum}$$

5. Weird Answers

Sometimes the optimal choice doesn't exist

- Here's a problem with no (finite) solution:
  - A soybean farm faces a competitive price: $p = 20$
  - Constant marginal cost of production: $C(q) = 18q$
- What should the firm do?
  - Profit function: $\pi(q) = 20q - 18q = 2q$
  - Set $q = \infty \implies$ make infinite profit
  - So there is no well-defined solution

Sometimes there are lots of optimal choices

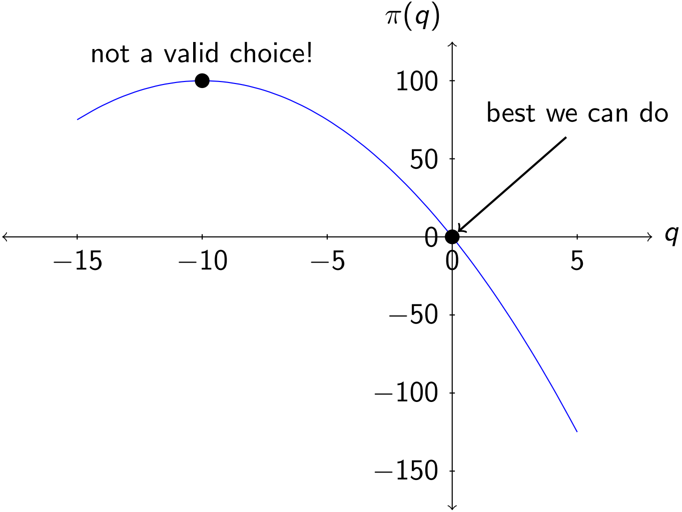
- Let's change the problem slightly:
  - Competitive price: $p = 20$
  - Cost of production: $C(q) = 20q$
- What should the firm do?
  - Profit function: $\pi(q) = 20q - 20q = 0$
  - Whatever q it chooses, it makes zero profit
  - So every choice of q is an optimal choice

And sometimes it's a "corner solution"

- Suppose that $p = 20$ and that costs are $C(q) = 40q + q^2$
  - Profit function: $\pi(q) = 20q - 40q - q^2 = -20q - q^2$
  - FOC: $-20 - 2q = 0 \implies q^* = -10$
  - But the firm can't produce negative output!
- So, what should the firm do?
  - Marginal profit: $\frac{d\pi(q)}{dq} = -20 - 2q < 0$
  - The more it produces, the more money it loses
  - So it should produce as little as possible: $q^* = 0$

A corner solution

Equation: $\pi(q) = -20q - q^2$



Corner solutions vs. interior solutions

- How many hours should I sleep each day?
  - Choice set: h must be in the interval $[0, 24]$
  - Corner solution: choosing $h^* = 0$ or $h^* = 24$
  - Interior solution: choosing h^* for which $0 < h^* < 24$
- Corner solutions may seem like "trick questions"
- But people and firms are at corner solutions all the time
  - I own zero Ferraris ☹
  - Apple has stopped selling the iPhone 7
  - Many CVS pharmacies are open 24 hours per day
- So we will see them a lot in this course

6. Two Choice Variables

Optimization with two choice variables

- So far, we've been analyzing problems with <u>one</u> choice variable
- In the real world, we often make multiple interrelated decisions
  - Should I go to college? If so, what should I major in?
  - How many iPhones should Apple make, and how many iPads?
- Suppose we have two choice variables, say q_1 and q_2
  - We indicate this by writing "$\max\limits_{q_1, q_2}$" instead of "$\max\limits_{q}$"
  - We will have *one first-order condition for each choice variable*
  - This gives us a system of two equations in two unknowns
  - We can solve this system of equations to obtain q_1^* and q_2^*

Example: a two-crop farm

Step 1: write down the optimization problem

- A farm sells a mixture of corn and squash at competitive prices
  - Price of corn (good 1): $p_1 = 15$
  - Price of squash (good 2): $p_2 = 12$
  - Cost of production: $C(q_1, q_2) = q_1^2 + q_2^2 + q_1 q_2$
- We write the optimization problem as:

$$\max_{q_1, q_2} \pi(q_1, q_2) = \underbrace{15q_1 + 12q_2}_{\text{total revenue}} - \underbrace{q_1^2 - q_2^2 - q_1 q_2}_{\text{total costs}}$$

Example: a two-crop farm

Step 2: take the first-order conditions

- Here is the problem again:

$$\max_{q_1, q_2} \pi(q_1, q_2) = 15q_1 + 12q_2 - q_1^2 - q_2^2 - q_1 q_2$$

- Next, we *partially differentiate* profits with respect to q_1
  - Notation for partial derivatives: $\frac{\partial}{\partial q_1}\pi(q_1, q_2)$
  - Put simply: take the derivative while holding q_2 constant

- Taking the FOCs for goods 1 and 2, respectively:

$$\frac{\partial}{\partial q_1}\pi(q_1, q_2) = 15 - 2q_1^* - q_2^* = 0$$
$$\frac{\partial}{\partial q_2}\pi(q_1, q_2) = 12 - 2q_2^* - q_1^* = 0$$

Example: a two-crop farm

Step 3: solve the system of equations

- We now have a system of 2 equations in 2 unknowns:

$$15 - 2q_1^* - q_2^* = 0$$
$$12 - 2q_2^* - q_1^* = 0$$

- Isolate q_2^* in the first equation:

$$q_2^* = 15 - 2q_1^*$$

- Then plug this into the second equation and solve:

$$12 - 2(15 - 2q_1^*) - q_1^* = 0 \implies q_1^* = 6$$
$$\implies q_2^* = 15 - 2(6) = 3$$

7. Comparative Statics

Learning about causal relationships

- Lots of important questions are about **causal relationships**
  - How would putting tariffs on Chinese goods affect US GDP?
  - How would a tax on Juul affect cigarette consumption?
- To answer these questions, we perform **comparative statics**
  - Express market outcomes as a function of parameters
  - See how outcomes change when the parameters change

Example: corn and squash

- Suppose we want to know:
  - If the price of corn falls, will our farm grow more squash?
  - Will it stop growing corn?
- One approach: experiment with specific values
  - Re-do the problem with $p_1 = 12$ instead of $p_1 = 15$
  - We'd find that q_1^* falls from 6 to 4 and q_2^* rises from 3 to 4
- But this approach does not work very well
  - May have to repeat this procedure multiple times
  - And it's hard to interpret the results
- Instead: solve the problem with p_1 and p_2 unspecified

Example: corn and squash

1. Rewrite the problem in terms of p_1 and p_2:

$$\max_{q_1,q_2} \pi(q_1, q_2) = p_1 q_1 + p_2 q_2 - q_1^2 - q_2^2 - q_1 q_2$$

2. Take the two FOCs:

$$\frac{\partial}{\partial q_1}\pi(q_1, q_2) = p_1 - 2q_1^* - q_2^* = 0$$
$$\frac{\partial}{\partial q_2}\pi(q_1, q_2) = p_2 - 2q_2^* - q_1^* = 0$$

Example: corn and squash

3. Solve for q_1^* and q_2^*:

$$q_1^*(p_1, p_2) = \frac{2}{3}p_1 - \frac{1}{3}p_2$$
$$q_2^*(p_1, p_2) = \frac{2}{3}p_2 - \frac{1}{3}p_1$$

4. Need to make sure these are valid choices!
   - For $q_1^* \geq 0$, we need $\frac{2}{3}p_1 - \frac{1}{3}p_2 \geq 0 \implies p_1 \geq \frac{1}{2}p_2$
   - For $q_2^* \geq 0$, we need $\frac{2}{3}p_2 - \frac{1}{3}p_1 \geq 0 \implies p_2 \geq \frac{1}{2}p_1$

5. Now we can do comparative statics:
   - Squash output is increasing in price of squash: $\frac{\partial q_2^*}{\partial p_2} = \frac{2}{3} > 0$
   - Squash output is decreasing in price of corn: $\frac{\partial q_2^*}{\partial p_1} = -\frac{1}{3} < 0$
   - If $p_1 < \frac{1}{2}p_2$, the farm stops growing corn $(q_1^* = 0)$

Next up

- Discussion section: perfect competition, effects of a tax
- Thursday 10/03: monopoly pricing (Lecture Note 3)
- Friday 10/04: Homework #1 due