Lecture Note 8: Static Games with Mixed Strategies

Lisa and Bart Simpson are playing **Rock-Paper-Scissors**:

We can represent the game as follows:

		Bart		
		Rock	Paper	Scissors
Lisa	Rock	$0, 0$	$-1, \textcircled{1}$	$\textcircled{1}, -1$
	Paper	$\textcircled{1}, -1$	$0, 0$	$-1, \textcircled{1}$
	Scissors	$-1, \textcircled{1}$	$\textcircled{1}, -1$	$0, 0$

(Remember from last time: to indicate each player's best response to each opposing strategy, I circle the payoff corresponding to the best response.)

Contrary to Bart's reasoning, Rock isn't really a dominant strategy. In fact this game doesn't even have a pure strategy Nash equilibrium (PSNE). Once Bart learns some game theory, how should we expect him to play?

Mixed strategies and expected payoffs

A **mixed strategy** is a decision to pick one of the available actions at random, assigning some probability to each action.

- Example: "Rock" with probability $\frac{4}{5}$, "Paper" with probability $\frac{1}{5}$.
- Example: flip a coin; if heads, kick the ball left; if tails, kick right.

A player decides how much probability "weight" to put on each option.

- Every probability must be between 0 and 1.
  - Players can set $p = 0$ for some strategies.
    (If a strategy is strictly dominated, it'll never get played.)
  - Players can set $p = 1$ for one strategy.
    (Technically, pure strategies count as mixed strategies too.)
- The probabilities must add up to 1.

We assume that players try to maximize their **expected payoff**.
The expected payoff is the expected value ("mean") of the payoff.

- An expected value is a weighted average of the possible outcomes.
  I'll write it with a funny-looking "E", for "expected": $\mathbb{E}(\text{payoff})$.
- If I receive x_1 with probability p_1, x_2 with probability p_2, etc.,

$$\mathbb{E}(\text{payoff}) = p_1 x_1 + p_2 x_2 + \ldots p_N x_N$$

Example: 25% chance of getting \$40, 75% chance of getting \$60.

$$\mathbb{E}(\text{payoff}) = \underbrace{25\% \times \$40}_{=\$10} + \underbrace{75\% \times \$60}_{=\$45} = \$55$$

<u>How not to lose at Rock-Paper-Scissors</u>

		Bart		
		Rock	Paper	Scissors
Lisa	Rock	$0, 0$	$-1, \textcircled{1}$	$\textcircled{1}, -1$
	Paper	$\textcircled{1}, -1$	$0, 0$	$-1, \textcircled{1}$
	Scissors	$-1, \textcircled{1}$	$\textcircled{1}, -1$	$0, 0$

Suppose Lisa thinks there's a $\frac{1}{2}$ chance Bart will play Rock, a $\frac{1}{4}$ chance he'll play Paper, and a $\frac{1}{4}$ chance he'll play Scissors.

- If she plays Rock: $\mathbb{E}(\text{payoff}) = \frac{1}{2} \cdot 0 + \frac{1}{4} \cdot -1 + \frac{1}{4} \cdot 1 = 0$.
- If she plays Paper: $\mathbb{E}(\text{payoff}) = \frac{1}{2} \cdot 1 + \frac{1}{4} \cdot 0 + \frac{1}{4} \cdot -1 = 0.25$.
- If she plays Scissors: $\mathbb{E}(\text{payoff}) = \frac{1}{2} \cdot -1 + \frac{1}{4} \cdot 1 + \frac{1}{4} \cdot 0 = -0.25$.

So Lisa should play Paper.

What if Lisa thinks there's a $\frac{1}{3}$ chance Bart plays each strategy?

- No matter what she picks, Lisa's expected payoff is zero.
- Lisa is therefore <u>indifferent</u> among all her options.
- Since every action is a best response, she would be willing to pick one at random: for example, with $\frac{1}{3}$ chance of each action. (By "willing", I mean that if somebody told her to, she would not object.)
- By the same logic: if Bart expects Lisa to pick each action $\frac{1}{3}$ of the time, then he is willing to adopt the same exact strategy.

If both of them play like this, neither will outperform the other on average, and neither will have any incentive to change their behavior.

Mixed strategy Nash equilibrium

Let's extend our earlier definitions of best responses and Nash equilibria:

- A mixed strategy is a **best response** if it maximizes a player's expected payoff given the strategy chosen by the other player.
- A **mixed strategy Nash equilibrium (MSNE)** is a pair of mixed strategies such that each player is playing a best response against the other player's (mixed) strategy.

In Rock-Paper-Scissors:

- There is no pure strategy Nash equilibrium.
- But there is a mixed strategy equilibrium in which each player chooses each action with probability $\frac{1}{3}$.

Mixed strategy Nash equilibria can help us think about situations in which the players might want to be "unpredictable".

- Penalty kicks: sometimes kick left, sometimes kick right.
(If I always kick left, the goalie will know where to go.)
- Foreign policy: sometimes be aggressive, sometimes not.
(If I'm never aggressive, you'll take advantage of me.)

We've seen that some games have no pure strategy Nash equilibria.
So how do we know whether we can find a mixed strategy equilibrium?

- John Nash proved that every game with only a finite number of strategies has at least one (pure or mixed) Nash equilibrium.
- Games with an infinite number of strategies (like "Name the Biggest Number") may have no Nash equilibria whatsoever.

Example: pharmaceutical entry

Consider the following **entry game**:

- Two pharmaceutical companies, Pfizer and Novartis, are deciding whether to develop (competing) drugs to treat opioid withdrawal.
- If only one company enters the market, it makes a lot of money.
- If both enter, they split the market and can't cover the R&D costs.

Here's the payoff matrix:

		Novartis: Enter	Novartis: Pass
Pfizer	Enter	$-1, -1$	⓷, ⓪
	Pass	⓪, ⓷	$0, 0$

This game has two PSNE: (Enter, Pass) and (Pass, Enter).

But there is something a little unrealistic about these equilibria:

- Pfizer and Novartis are identical (symmetric) here.
- But in each PSNE, they behave very differently.
- Why would Pfizer "give up"? Why would Novartis?

A mixed strategy equilibrium may be a more realistic outcome.

Exercise: Suppose that each firm chooses Enter with probability $\frac{3}{4}$ and Pass with probability $\frac{1}{4}$. Show that this pair of strategies is a mixed strategy Nash equilibrium. What is each firm's equilibrium payoff?

Answer: With these probabilities, each firm is indifferent between its two actions: for example, Pfizer's expected payoff from "Enter" equals $\frac{3}{4} \times -1 + \frac{1}{4} \times 3 = 0$, and its expected payoff from "Pass" equals $\frac{3}{4} \times 0 + \frac{1}{4} \times 0 = 0$. Therefore, this is a MSNE. The equilibrium payoffs are (0, 0).

Appendix: how to find a mixed strategy Nash equilibrium

Last year, I taught students how to find MSNEs "from scratch". But explaining this takes a lot of time, and we have a lot of material left to cover, so I've decided to skip it this year. I am including the details here in case you are curious, but I won't ask you to do this on exams.

Let's look for a mixed strategy Nash equilibrium in which Pfizer enters with probability α and Novartis enters with probability β.

		Novartis	
		Enter	Pass
Pfizer	Enter	$-1, -1$	$\textcircled{3}, \textcircled{0}$
	Pass	$\textcircled{0}, \textcircled{3}$	$0, 0$

Key insight: in any MSNE, both players must be <u>indifferent</u> among any strategies that they play with positive probability.

- To find <u>Pfizer's</u> probability of entry (α), make <u>Novartis</u> indifferent:
  - If Novartis enters: $\mathbb{E}(\text{payoff}) = \alpha(-1) + (1-\alpha)(3) = 3 - 4\alpha$.
  - If Novartis passes: $\mathbb{E}(\text{payoff}) = \alpha(0) + (1-\alpha)(0) = 0$.
  - Novartis is indifferent if: $3 - 4\alpha = 0 \implies \alpha^* = \frac{3}{4}$.
- To find <u>Novartis</u>'s probability of entry (β), make <u>Pfizer</u> indifferent:
  - If Pfizer enters: $\mathbb{E}(\text{payoff}) = \beta(-1) + (1-\beta)(3) = 3 - 4\beta$.
  - If Pfizer passes: $\mathbb{E}(\text{payoff}) = \beta(0) + (1-\beta)(0) = 0$.
  - Pfizer is indifferent if: $3 - 4\beta = 0 \implies \beta^* = \frac{3}{4}$.

So this game has an MSNE in which each firm enters with probability $\frac{3}{4}$ and passes with probability $\frac{1}{4}$. (Since the firms are identical, it makes sense that they'd follow the same strategy in the MSNE.)

Example: a predator-prey game

A wolf is hunting a deer. The deer can hide in the woods or in a field, and the wolf can look for the deer in either location:

		Wolf	
		Woods	Field
Deer	Woods	1, (1)	(3), 0
	Field	(3), 0	0, (3)

Suppose Deer picks Woods with probability α. Let's make Wolf indifferent.

- If Wolf plays Woods: $\mathbb{E}(\text{payoff}) = \alpha(1) + (1-\alpha)0 = \alpha$.
- If Wolf plays Field: $\mathbb{E}(\text{payoff}) = \alpha(0) + (1-\alpha)3 = 3 - 3\alpha$.
- Wolf is indifferent if: $\alpha = 3 - 3\alpha \implies \alpha^* = \frac{3}{4}$.

Suppose Wolf picks Woods with probability β. Let's make Deer indifferent.

- If Deer plays Woods: $\mathbb{E}(\text{payoff}) = \beta(1) + (1-\beta)3 = 3 - 2\beta$.
- If Deer plays Field: $\mathbb{E}(\text{payoff}) = \beta(3) + (1-\beta)0 = 3\beta$.
- Deer is indifferent if $3 - 2\beta = 3\beta \implies \beta^* = \frac{3}{5}$.

Thus, there is a mixed strategy Nash equilibrium in which

- Deer picks Woods with probability $\frac{3}{4}$, Field with prob. $\frac{1}{4}$.
- Wolf picks Woods with probability $\frac{3}{5}$, Field with prob. $\frac{2}{5}$.

This is the <u>only</u> Nash equilibrium. If the probabilities were anything else, at least one player could increase $\mathbb{E}(\text{payoff})$ by doing something different.

