Lecture Note 13: Making Decisions in an Uncertain World

We've seen how people make decisions when they know what'll happen.

- Consumers pick the bundle that maximizes utility.
- Firms use backward induction to decide whether to enter a market.

We should all be so lucky!

Uncertainty is an inherent part of the human experience:

- When you go to see a movie, you don't know if you'll like it.
- When Pfizer tests a new drug, it doesn't know if the drug will work.
- When you invest in the stock market, you're taking on risk.
- When you save for retirement, you don't know how long you'll live.
- When you buy a used car, it might turn out to be a "lemon".

Big question: how do people make decisions in the presence of uncertainty?

In this lecture note, we'll look at:

- How to measure risk
- How to describe people's attitudes towards risk
- How to choose the best option in a risky situation

In the following lecture note, we'll talk about the strategies that individuals and firms use to reduce their exposure to risk.

A crash course in probability

An **event** is something that might happen.

- Rolling a 2 on a six-sided die.
- Rolling an even number on a six-sided die.
- Getting into a car accident.

A **probability** is a number between 0 and 1 indicating the likelihood that a given event occurs (0 = never happens, 1 = happens for sure).

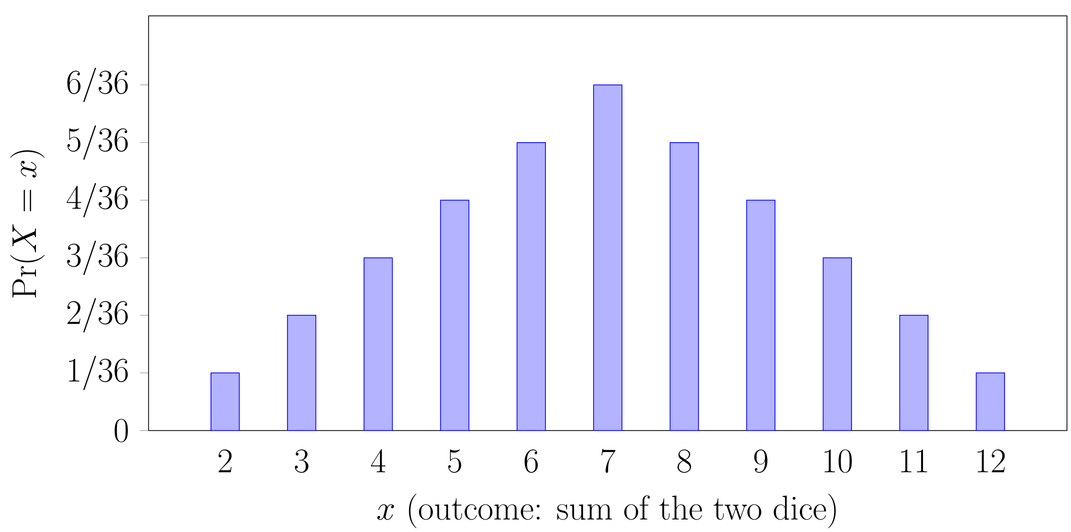
- $\Pr(\text{roll a 2 on a six-sided die}) = \frac{1}{6}$.
- $\Pr(\text{roll an even number on a six-sided die}) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$.

A **random variable** is a quantity whose value is uncertain.

- Example: the price of 1 share of Apple stock next Monday at 3:00pm.

A **probability distribution** is a function that tells us the probability that a random variable will take on any given value.

Example: You roll two dice. Let X be the sum of their values.
What is the probability distribution of X?*



*There are 36 possible pairs of numbers, e.g., rolling a 4 on the first die and a 5 on the second (in which case $X = 9$). We construct each probability by counting the number of ways to generate that outcome: e.g., there are two ways to get $X = 3$—rolling (1, 2) or rolling (2, 1)—so $\Pr(X = 3) = \frac{2}{36}$.

Expected value and variance

Suppose that you're deciding whether to invest in a bond or a stock:

- Bond is worth \$15 with probability 1.
- Stock is worth \$5 with probability $\frac{1}{3}$ and \$20 with probability $\frac{2}{3}$.

Before investing, you'd want to know how much each asset is worth on average (its expected value, or "mean") and how risky it is (its variance).

Let X be a random variable that equals x_1 with probability p_1, x_2 with probability p_2, ..., and x_N with probability p_N.

The **expected value** of X is a weighted average of its possible values:

$$\mathbb{E}(X) = p_1 x_1 + p_2 x_2 + \ldots p_N x_N \quad \text{or, equivalently,} \quad \mathbb{E}(X) = \sum_{i=1}^{N} p_i x_i.$$

Let X_B denote the value of the bond and X_S the value of the stock.

- What is $\mathbb{E}(X_B)$? $= 1 \cdot 15 = 15$.
- What is $\mathbb{E}(X_S)$? $= \frac{1}{3} \cdot 5 + \frac{2}{3} \cdot 20 = 15$.

The **variance** of X measures the "spread" of its probability distribution.†

$$\text{Var}(X) = \sum_{i=1}^{N} p_i (x_i - \mathbb{E}(X))^2$$

The **standard deviation** is the square root of the variance.

- What is $\text{Var}(X_B)$? $= 1 \cdot (15 - 15)^2 = 0$.
- What is $\text{Var}(X_S)$? $= \frac{1}{3} \cdot (5 - 15)^2 + \frac{2}{3} \cdot (20 - 15)^2 = 50$.
- What is $\text{SD}(X_S)$? $= \sqrt{50} \approx 7.07$.

†It's sometimes easier to calculate the variance using another formula, $\text{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$. You can use whichever formula you prefer.

Expected-utility maximization

When the world is certain, we assume that:

- People choose the option that maximizes their utility, $u(x)$.
- Firms choose the option that maximizes their profits, $\pi(x)$.

When the world is *uncertain*, we assume that:

- People try to maximize their **expected utility**, $\mathbb{E}(u(x))$.
- Firms try to maximize their **expected profit**, $\mathbb{E}(\pi(x))$.

(We've seen this idea before: in games with mixed strategies, players try to maximize their *expected payoff*.)

Example 1: $u(x) = \ln(x)$, where ln is the natural logarithm.

- Bond: $\mathbb{E}(u(x)) = 1 \cdot \ln(15) \approx 2.71$.
- Stock: $\mathbb{E}(u(x)) = \frac{1}{3} \cdot \ln(5) + \frac{2}{3} \cdot \ln(20) \approx 2.53$.
- Optimal choice: invest in the bond.

Example 2: $u(x) = x$.

- Bond: $\mathbb{E}(u(x)) = 1 \cdot 15 = 15$.
- Stock: $\mathbb{E}(u(x)) = \frac{1}{3} \cdot 5 + \frac{2}{3} \cdot 20 = 15$.
- Optimal choice: invest in either asset (agent is indifferent).

Example 3: $u(x) = x^2$.

- Bond: $\mathbb{E}(u(x)) = 1 \cdot 15^2 = 225$.
- Stock: $\mathbb{E}(u(x)) = \frac{1}{3} \cdot 5^2 + \frac{2}{3} \cdot 20^2 = 275$.
- Optimal choice: invest in the stock.

When we analyze choice without uncertainty, utility functions tell us about people's preferences—or "tastes"—for different goods.

Here, the utility function tells us about people's taste or distaste for risk.

Attitudes towards risk

To understand how people respond to risk, we need a way to classify their attitudes towards risk. There are many ways to do this. Here's one:

We say that a transaction is a **fair bet** if its expected value is zero. Example: paying $15 to buy the stock discussed on the previous page.

If you play a fair bet many times, you'll break even on average.

We can use the idea of a fair bet to describe people's risk attitudes:

- A **risk-averse** person doesn't want to take a fair bet.
- A **risk-neutral** person is indifferent towards a fair bet.
- A **risk-loving** person always wants to take a fair bet.

Risk attitudes are closely related to the shape of the utility function:

- Risk-averse agents have *concave* utility functions.
- Risk-neutral agents have *linear* utility functions.
- Risk-loving agents have *convex* utility functions.

Recall from calculus that a function $u(x)$ is concave if its second derivative $u''(x)$ is negative, convex if $u''(x)$ is positive, and linear if $u''(x) = 0$.

- Example 1: $u(x) = \ln(x) \implies u'(x) = \frac{1}{x}, u''(x) = -\frac{1}{x^2} < 0$.
- Example 2: $u(x) = x \implies u'(x) = 1, u''(x) = 0$.
- Example 3: $u(x) = x^2 \implies u'(x) = 2x, u''(x) = 2 > 0$.

These functions are, respectively, concave, linear, and convex. Notice that the person with concave utility preferred the bond, the one with convex utility preferred the stock, and the one with linear utility was indifferent.

Certainty equivalents

Suppose $u(x) = \sqrt{x}$. Is this person risk-averse, risk-neutral, or risk-loving?

- $u(x) = \sqrt{x} = x^{\frac{1}{2}} \implies u'(x) = \frac{1}{2}x^{-\frac{1}{2}}$, $u''(x) = -\frac{1}{4}x^{-\frac{3}{2}} < 0$.
- Since $u''(x) < 0$, $u(x)$ is concave, so this person is risk-averse.

Suppose she's deciding whether to buy a lottery ticket that gives her a payoff of $X = 0$ with probability $\frac{1}{3}$ and $X = 9$ with probability $\frac{2}{3}$.

- Expected value of the lottery ticket: $\mathbb{E}(X) = \frac{1}{3} \cdot 0 + \frac{2}{3} \cdot 9 = 6$.
- Expected utility from the lottery: $\mathbb{E}(u(X)) = \frac{1}{3} \cdot \sqrt{0} + \frac{2}{3} \cdot \sqrt{9} = 2$.
- The agent would get higher expected utility from getting \$6 for sure than she gets from a risky lottery worth \$6 on average:

$$u(6) \approx 2.45 > \mathbb{E}(u(X)) = 2$$

To put it another way, a risk-averse agent would never pay \$6 to gamble on a risky lottery that pays out \$6 on average.

Would a risk-averse agent *ever* play the lottery? Maybe, if the price of a lottery ticket is low enough.

An agent's **certainty equivalent** ($CE(X)$) is the guaranteed amount that she would view as equally desirable as a risky asset with (random) value X. So, she is indifferent between getting $CE(X)$ for sure or instead taking a gamble on the risky lottery ticket. The CE solves the equation

$$u(CE(X)) = \mathbb{E}(u(X))$$

We calculate the CE based on this indifference condition:

$$\sqrt{CE(X)} = 2 \implies CE(X) = 4$$

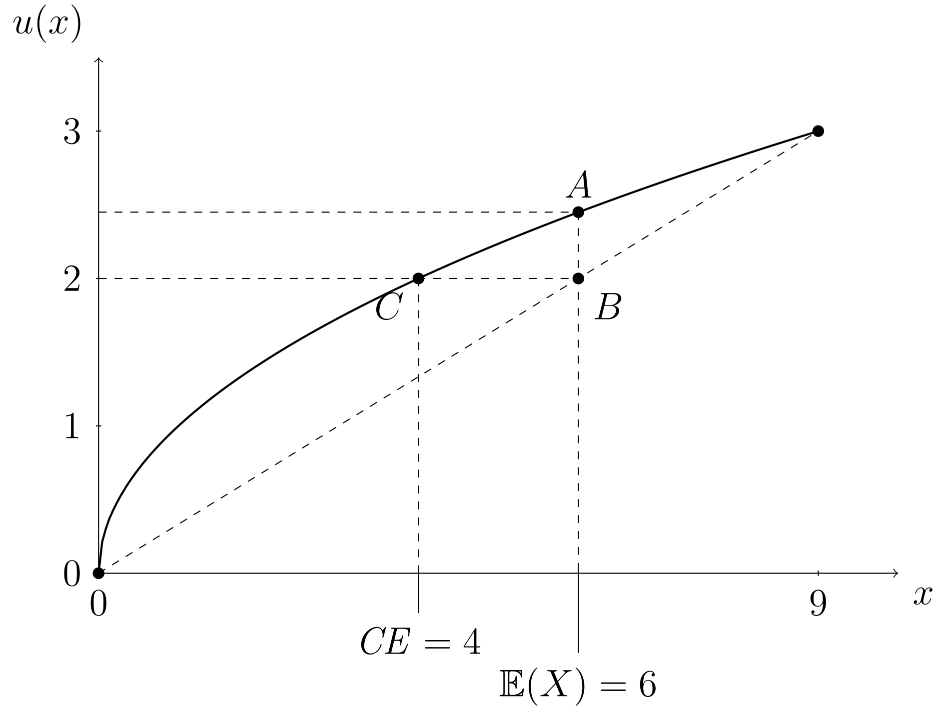
Our agent would be willing to pay \$4 to buy a ticket for this lottery.

Risk premiums (not covered on final exam)

The certainty equivalent is closely related to the **risk premium**, the amount of money that an agent who owns a risky asset would be willing to pay to eliminate this risk. The risk premium is the gap between the expected value of the asset and the agent's certainty equivalent:

$$RP(X) = \mathbb{E}(X) - CE(X)$$

In this case, $RP(X) = 6 - 4 = 2$. The agent would be willing to pay \$2 to eliminate the risk associated with the lottery ticket.‡



We can see this in the figure above, where I've plotted $u(x)$ against x:

- Point A shows the agent's utility from getting \$6 for sure.
- Point B shows the agent's (expected) utliity from playing the lottery.
- Point C shows the agent's utility from getting $CE(X)$ for sure.

The risk premium is the horizontal gap between points B and C.

‡For a risk-averse agent, the risk premium is positive: a risk-averse agent needs to be compensated before she's willing to accept a risk. For a risk-loving agent, it's negative: she'd gladly pay money to take on extra risk. For a risk-neutral agent, the risk premium is zero.

