Intermediate Microeconomic Theory
ECN 100B, Fall 2019
Professor Brendan Price

Section Problems #7
(Week of Monday, November 25)

*Note: There is no section this week: the questions below are purely for self-study. They are quite challenging, so don't be discouraged if you have trouble solving them. As always, I will be posting detailed solutions that walk you through how to solve these problems.*

# Brain teasers

Zombie Homer and Zombie Flanders share a flower garden between their neighboring graves.[1] The garden is a public good: either neighbor can do the work of watering and fertilizing the flowers, but both neighbors benefit from being able to sit and admire the flowers after dinner.

Let $Q = q_H + q_F$ denote total time spent gardening, where $q_H$ and $q_F$ are time spent by Homer and Flanders, respectively. Homer and Flanders receive private marginal benefits

$$p_H(Q) = 12 - Q$$
$$p_F(Q) = 12 - Q$$

Up to now, we've only seen problems where each player has a constant marginal cost of providing the public good. In many situations, however, it's more reasonable to think that the players have *increasing* marginal costs: in this problem, for example, the zombies might get more and more tired/bored/hungry the more time they spend gardening.

So, let's suppose that Homer and Flanders have total costs given by

$$C_H(q_H) = q_H^2$$
$$C_F(q_F) = q_F^2$$

## The Nash equilibrium

We can think of this problem as a static game. With an increasing marginal cost function, it turns out that this problem is mathematically a lot like the Cournot games we studied earlier in the course. So we'll use a "Cournot-style" approach to find the Nash equilibrium.[2]

[1] This problem is loosely inspired by *The Simpsons*, Treehouse of Horror III (link to Youtube clip).

[2] One could also study a dynamic version of this game in which Homer moves first and Flanders moves second. In that case, we'd use backward induction to find the game's subgame-perfect Nash equilibrium. The mathematics would be similar to what we saw in the Stackelberg game.

 

a. Suppose Homer expects Flanders to choose some amount $\hat{q}_F$. Find Homer's best-response function $q_H^* = BR_H(\hat{q}_F)$. (Hint: given his "guess", Homer will contribute up to the point where his private marginal benefit equals his private marginal cost.) You can assume there's an interior solution (the Nash equilibrium has $q_H^* > 0$ and $q_F^* > 0$).

Setting Homer's PMB equal to his PMC gives us

$$\underbrace{12 - q_H - \hat{q}_F}_{PMB_H} = \underbrace{2q_H}_{PMC_H} \implies q_H^* = BR_H(\hat{q}_F) = 4 - \frac{1}{3}\hat{q}_F$$

b. Now find Flanders's best-response function $q_F^* = BR_F(\hat{q}_H)$. (Hint: remember the "symmetry shortcut" we used in Cournot. Are the players here symmetric?)

The players are symmetric here, so Flanders's best-response function is the mirror image of Homer's:

$$\underbrace{12 - \hat{q}_H - q_F}_{PMB_F} = \underbrace{2q_F}_{PMC_F} \implies q_F^* = BR_F(\hat{q}_H) = 4 - \frac{1}{3}\hat{q}_H$$

c. Recall that, in a Nash equilibrium, both "guesses" must be right (otherwise, whoever guessed wrong wouldn't be playing a best response). Use this fact to turn the two best-response functions into a system of two equations in two unknowns ($q_H^*$ and $q_F^*$). Solve this system of equations to obtain the Nash equilibrium strategies $q_H^*$ and $q_F^*$. How many units of the public good are provided in this Nash equilibrium ($Q^{\text{Nash}}$)?

The two equations are

$$q_H^* = 4 - \frac{1}{3}q_F^*$$
$$q_F^* = 4 - \frac{1}{3}q_H^*$$

Solving this system of equations yields $q_H^* = q_F^* = 3$, so the equilibrium quantity of the public good is $Q^{\text{Nash}} = 6$.

## The social optimum

Now let's think about the quantities $q_H^s$ and $q_F^s$ that would maximize total surplus. When both players share the same, constant marginal cost, it's sufficient for us to indicate the *total* amount $Q^s$ that's socially optimal, since in that case the total surplus doesn't depend on how we split up the work between $q_H$ and $q_F$. With our current marginal cost curves, however, there is a single best way to split the work between Zombie Homer and Zombie Flanders. So we'll need to be precise about the socially optimal values of $q_H^s$ and $q_F^s$ specifically.

With these marginal cost curves, it's tricky (though possible) to think in terms of "social marginal benefit = social marginal cost", because it's not immediately clear what the social marginal cost is. Instead, let's think about the social planner's optimization problem—how to choose $q_H$ and $q_F$ to maximize the total surplus:

$$\max_{q_H, q_F} \underbrace{SB(q_H + q_F)}_{\text{total social benefit}} \underbrace{- q_H^2 - q_F^2}_{\text{total social cost}}$$

d. Calculate the social marginal benefit curve, $SMB(Q)$. Then find the *total* social benefit $SB(Q)$ by calculating the area under $SMB(Q)$ between zero and $Q$.[3] Rewrite this as $SB(q_H + q_F)$ by replacing each $Q$ with $q_H + q_F$.

The social marginal benefit is $SMB(Q) = 24 - 2Q$. The integral is $SB(Q) = \int_0^Q (24 - 2x)dx = 24Q - Q^2$.

e. Substitute your expression $SB(q_H + q_F)$ into the social planner's optimization problem. Solve the optimization problem (taking FOCs and solving the system of equations) to determine $q_H^s$ and $q_F^s$. How many units of the public good *should* be provided ($Q^s$)? How does the social optimum compare to the Nash equilibrium quantity ($Q^{\text{Nash}}$)?

The social planner's problem is

$$\max_{q_H, q_F} 24(q_H + q_F) - (q_H + q_F)^2 - q_H^2 - q_F^2$$

There are two choice variables, so we get two FOCs:

$$24 - 2q_H - 2q_F - 2q_H = 0$$
$$24 - 2q_H - 2q_F - 2q_F = 0$$

Solving this system of equations, we first find that $q_H^s = q_F^s$. This makes sense: given the increasing marginal cost, the social planner wants to equalize the workload between the two zombies. We can then substitute this into one of the FOCs to obtain $q_H^s = q_F^s = 4$, so that $Q^s = 8$. Notice that $Q^{\text{Nash}} < Q^s$: as usual, the public good is underprovided.

[3] I try to avoid using integrals in this class, but taking the integral $SB(Q) = \int_0^Q SMB(x)dx$ is the easiest way to do this. An alternative approach is to draw the graph, pick some value of $Q$, divide the area under the curve into a rectangle and a triangle, calculate each shape's area as a function of $Q$, and add them up.

 