Intermediate Microeconomic Theory
ECN 100B, Fall 2019
Professor Brendan Price

Section Problems #8
(Week of Monday, December 2)

I could've sworn I parked here . . .

Consider a Davis student whose utility function is $u(w) = \sqrt{w}$, where $w \geq 0$ is her wealth, which equals her financial wealth w_f plus the value of her bike w_b:

$$w = w_f + w_b$$

The student starts off with \$144 in cash ($w_f = 144$) and owns a bike worth \$81 ($w_b = 81$). There is a 1/3 chance the student's bike is stolen. Otherwise, it's fine.

a. Is this student risk-averse, risk-neutral, or risk-loving? Prove this mathematically.

   She's risk-averse. To see this, differentiate the utility function twice:

   $$u(w) = w^{\frac{1}{2}} \implies u'(w) = \frac{1}{2}w^{-\frac{1}{2}} \implies u''(w) = -\frac{1}{4}w^{-\frac{3}{2}}$$

   Since $u''(w) < 0$, the utility function is concave, which corresponds to risk aversion.

b. Compute each of the following:

   (i) The expected loss from bike theft (i.e., the expected value of the wealth she loses).

   This equals the probability of a bike theft times the financial loss if it's stolen:

   $$\text{expected loss} = \frac{1}{3} \times 81 = \$27$$

   (ii) The student's expected utility.

   The student's wealth is either \$144 (if the bike is stolen) or \$225 (if not). Expected utility is a weighted average of the utility she gets in each of these cases:

   $$\text{expected utility} = \mathbb{E}(u(w)) = \frac{1}{3}\sqrt{144} + \frac{2}{3}\sqrt{225} = 4 + 10 = 14$$

(iii) The student's certainty equivalent. (An agent is indifferent between playing a "lottery" and getting the corresponding certainty equivalent for sure.)

The certainty equivalent satisfies the condition $u(\text{CE}) = \mathbb{E}(u(w))$. That gives us

$$\sqrt{\text{CE}} = 14 \implies \text{CE} = 196$$

In words, the student would be indifferent between getting \$196 for sure vs. playing a lottery that gives her \$144 with probability $\frac{1}{3}$ and \$225 with probability $\frac{2}{3}$.

(iv) The student's risk premium. (The risk premium is the amount she'd be willing to give up to get her expected wealth for sure. It equals expected wealth minus the certainty equivalent.)

Expected wealth is $\mathbb{E}(w) = \frac{1}{3} \cdot 144 + \frac{2}{3} \cdot 225 = 198$. Subtracting the certainty equivalent, CE = 196, gives a risk premium of \$2.

c. Kryptonite's "New York" bike lock is supposed to keep your bike safe even on the mean streets of Manhattan. Suppose that purchasing the New York lock completely eliminates the risk of bike theft.

(i) How much would this student be willing to pay for the New York lock?

The student is indifferent between buying the lock and not buying the lock when

$$u(w_f + w_b - p^*) = \mathbb{E}(u(w))$$

where p^* is her willingness to pay (her "cutoff" price). Plugging in $w_f = 144$ and $w_b = 81$, and setting $\mathbb{E}(u(w)) = 14$ (which we calculated in part b(ii)), gives us

$$\sqrt{225 - p^*} = 14 \implies p^* = 29$$

So the agent would be willing to pay up to \$29 for a New York lock. Notice that her willingness to pay equals the expected loss (\$27) plus the risk premium (\$2). If she were risk neutral, her willingness to pay would exactly equal the expected loss from bike theft. Since she's risk-averse, she's willing to pay a little extra (the risk premium) to eliminate the risk of theft.

(ii) Now suppose that the student has just won a Powerball lottery, increasing her financial wealth to $w_f = 1600$. Compute her new willingness to pay for the lock.

We find the new willingness to pay by finding the price that makes the student indifferent between buying a lock and risking having her bike stolen (which results in total wealth of either 1600 or 1681 depending on whether she loses her bike):

$$\sqrt{1681 - p^*} = \frac{1}{3}\sqrt{1600} + \frac{2}{3}\sqrt{1681} = 40.67 \implies p^* = 27.22$$

This new WTP is quite close to the expected loss from bike theft. In fact this student's risk premium is just \$0.22, much smaller than it was before. The intuition for this result is that, at higher wealth levels, the marginal utility of wealth is much lower, which makes the student care less about modest fluctuations in her wealth in this range.

(iii) The results above suggest that richer students may have *lower* willingness to pay for a bike lock. Can you think of a reason why richer students might have *higher* willingness to pay? (I can think of two main reasons.)

We've been assuming that the student rides the same bike before and after winning the Powerball. In general, however, we'd expect wealthier students to ride more expensive bicycles, so that w_b is greater for wealthier students. The nicer the bike, the more you'll be willing to pay for a good lock, so rich students might have higher willingness to pay because they ride nicer bikes.

Another possibility is that poor students may be *credit constrained*: if w_f is low enough, and if it's difficult to borrow (e.g., credit cards are already maxed out), a student simply may not have enough money to afford an expensive bike lock. In that case, even if everyone rides equally valuable bikes, we might observe that only rich students buy good locks.

Playing the stocks

There are two stocks, A and B. One share of either stock is worth \$30 with probability $1/2$ and it's worth \$50 with probability $1/2$.

a. Compute the expected value and variance of a share of stock A.

Using the formulas I gave you in class,

- expected value $= \frac{1}{2} \times 30 + \frac{1}{2} \times 50 = 40$
- variance $= \frac{1}{2} \times (30 - 40)^2 + \frac{1}{2} \times (50 - 40)^2 = 100.$

b. Compute the expected value of a portfolio consisting of one share of A and one of B.

Since each of the two shares has an expected value of \$40, any two-share portfolio has an expected value of \$80.

c. Compute the variance of this portfolio under each of the following assumptions:

(i) The shares are perfectly negatively correlated: whenever stock A performs well (equaling \$50), stock B performs badly (equaling \$30), and vice versa.

In this case, it's guaranteed that one stock will be worth \$30 and the other will be worth \$50, so the portfolio is worth its expected value of \$80 with probability 1. The variance is zero.

(ii) The shares are perfectly positively correlated: whenever stock A performs well, so does stock B; whenever stock A performs badly, so does stock B.

In this case, the portfolio is equally likely to be worth \$60 or \$100. The variance equals $\frac{1}{2} \times (60 - 80)^2 + \frac{1}{2} \times (100 - 80)^2 = 400$.

(iii) The shares are uncorrelated: their values are independent of one another.

Now there are three possible monetary outcomes. With probability $1/4$, both stocks are worth \$30, so the portfolio is worth \$60. With probability $1/2$, one is worth \$30 and the other is worth \$50, so the portfolio is worth \$80. With probability $1/4$, both stocks are worth \$50, so the portfolio is worth \$100. Putting it together, the variance equals $\frac{1}{4} \times (60-80)^2 + \frac{1}{2} \times (80-80)^2 + \frac{1}{4} \times (100-80)^2 = 200$.

d. Under which of these assumptions does diversification (buying one share of each stock) reduce the portfolio's variance, relative to buying two shares of the same stock?

A portfolio consisting of two shares of the same stock has a variance equal to 400. Diversifying eliminates the variance if the stocks are perfectly negatively correlated, and it reduces (but doesn't eliminate) the variance if they are uncorrelated. If the stocks are perfectly positively correlated, diversification has no effect on the variance.